SOME MODEL THEORETIC RESULTS FOR ω-LOGIC*

BY

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ABSTRACT

The ω-completeness theorem is applied to prove theorems above two-cardinal models, homogeneous models, and categoricity in power in ω-logic.

In this paper we shall apply the ω-completeness theorem to obtain results about two-cardinal models, homogeneous models, and categoricity in power. By ω-logic, L^ω, we mean the logic formed by adding to first order logic L a new unary predicate symbol N and individual constants $0,1,\cdots$. An *ω-model* is a model for L^ω in which $N = \{0,1,\cdots\}$. The ω-completeness theorem is a sufficient condition for a theory in ω-logic to have an ω-model; a statement of it and the relevant references are given in §1.

For an introduction, we shall give here a brief summary of our main results. Let K be the class of all ω-models of a theory T in ω-logic. If $\phi(x)$ is a formula and $\mathfrak{A}$ a model for L^ω, then $\phi(\mathfrak{A})$ is the set of all elements of $\mathfrak{A}$ which satisfy ϕ. In §2 we prove the following two-cardinal result: *If K has a model $\mathfrak{A}$ of power $\mathfrak{m}$ such that $\aleph_0 \leqq |\phi(\mathfrak{A})| < \mathfrak{m}$, then K has a model $\mathfrak{C}$ of power $\aleph_1$ such that $|\phi(\mathfrak{C})| = \aleph_0$.* The above theorem was proved by Vaught in [10] for the case where K is the class of all models of a theory in first order logic, and our result generalizes Vaught's theorem to ω-logic.

In §3 we use the results of §2 to prove: *If all models in K of power $\aleph_1$ are homogeneous, then all uncountable models in K are homogeneous.* This time the special case for first order logic appears to be new. (The definition and references for homogeneous models are given in §3).

In §4 we apply the results of §3 to prove that: *If K has a homogeneous model of power $\aleph_1$ and K is categorical in power $\aleph_1$, then K is categorical in every uncountable power.* For a theory T in first order logic, Morley [8] proved that: *if T is categorical in one uncountable power then T is categorical in every uncountable power.* It is known from Morley and Vaught [10] that, assuming the continuum hypothesis, every first order theory T which has infinite models has a homogeneous model of power $\aleph_1$. Thus, if we assume the continuum hypothesis, we have a new (and much shorter) proof of the upward part of Morley's

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theorem, and also a generalization of the upward part of Morley's theorem to ω-logic.

All our above results still work if K is a PC_δ^ω class, that is, the class of all reducts to L of ω-models of a theory in ω-logic with countably many extra predicates.

Besides the ω-completeness theorem we shall make extensive use of the results of Tarski and Vaught [15], and Morley and Vaught [10]. Some applications of the ω-completeness theorem to models of set theory will be given in [6]. The paper [7] contains theorems which are closely related to §3 and §4 of this paper, but do not use the ω-completeness theorem. Most of our results in this paper were announced in the abstract [5].

1. **Preliminaries.** We use the letters $\alpha, \beta, \cdots$ for ordinals, and $\mathfrak{m}, \mathfrak{n}, \cdots$ for cardinals. Cardinals are identified with initial ordinals. We shall work with a countable first-order predicate logic L with identity symbol. For the basic notions of model theory see, e.g. [13], [15]. Models are denoted by German capitals $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$, sometimes with subscripts, and the universe set of a model is denoted by the corresponding letters A, B, C. We shall sometimes enlarge the language L by adding new individual constant or predicate symbols. If a is an α-termed sequence of elements of a model $\mathfrak{A}$ for L, then $(\mathfrak{A}, a)$ is a model of the language $L(\alpha)$ formed by adding α new individual constants to L. Similarly, if R is an n-ary relation over A, then $(\mathfrak{A}, R)$ is a model for the logic $L(P)$ formed by adding a new n-ary predicate symbol P to L. The model $(\mathfrak{A}, R)$ is called an *expansion* of $\mathfrak{A}$ to a model for $L(P)$, and $\mathfrak{A}$ is called the *reduct* of $(\mathfrak{A}, R)$ to L. The symbols $\mathfrak{A} \equiv \mathfrak{B}$, $\mathfrak{A} \prec \mathfrak{B}$ mean that $\mathfrak{A}$ is elementarily equivalent to $\mathfrak{B}$, and $\mathfrak{A}$ is an elementary submodel of $\mathfrak{B}$. An *elementary chain* is a sequence

$$\mathfrak{A}_0 \prec \mathfrak{A}_1 \prec \cdots \prec \mathfrak{A}_\alpha \prec \cdots, \qquad \alpha < \beta,$$

of elementary extensions. The fundamental result about elementary chains, due to Tarski and Vaught [15], is that *the union* $\bigcup_{\alpha<\beta} \mathfrak{A}_\alpha$ *of an elementary chain is an elementary extension of each model* $\mathfrak{A}_\alpha$.

If T is a theory (set of sentences) in L, then a sentence ϕ is a *consequence* of T if every model of T is a model of ϕ. We say that ϕ is *consistent with* T if $T \cup \{\phi\}$ has a model. If $\phi(x)$ is a formula of L whose only free variable is x, then we let

$$\phi(\mathfrak{A}) = \{a \in A : a \text{ satisfies } \phi(x) \text{ in } \mathfrak{A}\}.$$

By *ω-logic* we shall mean the language L^ω obtained by adding to L a new unary predicate N and new individual constants $0,1,2,\cdots$, one for each natural number. A model

$$(\mathfrak{A}, N, 0, 1, 2, \cdots)$$

for L^ω is said to be an *ω-model* if $N = \{0,1,2,\cdots\}$. Two simple remarks about ω-models are:

Any submodel of an ω-model is an ω-model.

The union of a chain of ω-models is an ω-model.

A theory T in ω-logic is said to be *ω-complete* if

(1) $N(0)$, $N(1)$, $N(2)$, $\cdots$ are consequences of T;

(2) If $\phi(x)$ is a formula and $\phi(0), \phi(1), \phi(2), \cdots$ are consequences of T, then $\forall x(N(x) \to \phi(x))$ is a consequence of T.

The condition (2) is clearly equivalent to:

(2′) If $\phi(x)$ is a formula and $\exists x(N(x) \wedge \phi(x))$ is consistent with T, then there exists $n < \omega$ such that $\phi(n)$ is consistent with T.

The basic result about ω-logic which we shall use is the

ω-COMPLETENESS THEOREM. *Let T be a theory in ω-logic. If T has a model and is ω-complete, then T has an ω-model.*

Various forms of the ω-completeness theorem were obtained independently by several people, including Schutte, Henkin, Orey, Ryll-Nardzewski, (see Addison [1], p. 36 for references). The above formulation is due to Orey [17].

A class K of models for L is said to be a *pseudo-elementary class*, or a *PC_δ class*, if there is a finite or countable list of extra predicates $P_0, P_1, \cdots$ and a theory T in $L(P_0, P_1, \cdots)$ such that K is the class of all reducts to L of models of T. This notion is due to Tarski [13]. We shall say that K is an *ω-pseudo-elementary class*, or *PC_δ^ω class*, if there is a finite or countable sequence of extra predicates $P_0, P_1, \cdots$ and a theory T in $L^\omega(P_0, P_1, \cdots)$ such that K is the class of all reducts $\mathfrak{A}$ to L of *ω-models* $(\mathfrak{A}, N, 0, 1, \cdots, R_0, R_1, \cdots)$ of T.

It is clear that if T is a theory in L then the class of all models of T is a PC_δ class, but not conversely. Also, every PC_δ class is a PC_δ^ω class. The notion of a PC_δ^ω class is more general than it might first appear. For instance, consider the infinitary logic $L_{\omega_1\omega}$ which has all the symbols of L plus countably infinite conjuctions and disjunctions. Scott [12] has shown that if θ is a sentence in $L_{\omega_1\omega}$, then the class K of all models of θ is a PC_δ^ω class. In particular, if T is a theory in L, Σ is a set of formulas $\sigma(x)$, and K is the class of all models $\mathfrak{A}$ of T such that no element of $\mathfrak{A}$ satisfies all formulas of Σ, then K is a PC_δ^ω class. Likewise, if T is a set of sentences in the weak second order logic of Tarski [14], or a set of sentences in the logic with the extra quantifier "there exist infinitely many" (see Fuhrken [3]), then the class K of all models of T is a PC_δ^ω class. Finally, if L already has the symbols $N, 0, 1, \cdots$, then the class of all ω-models of a theory T in L is a PC_δ^ω class.

2. Löwenheim Skolem theorems for two cardinals. Vaught in [10] proved the following theorem:

Suppose $\mathfrak{A}$ *is a model for* L, $\phi(x)$ *is a formula, and* $\aleph_0 \leqq |\phi(\mathfrak{A})| < |A|$. *Then there is a model* $\mathfrak{C} \equiv \mathfrak{A}$ *such that* $|\phi(\mathfrak{C})| = \aleph_0$ *and* $|C| = \aleph_1$.

Chang conjectured that Vaught's theorem could be improved by replacing $\mathfrak{C} \equiv \mathfrak{A}$ by $\mathfrak{C} \prec \mathfrak{A}$. Rowbottom [11], however, showed that Chang's conjecture contradicts the axiom of constructibility. Our first theorem is an improvement of Vaught's theorem in the direction of Chang's conjecture.

THEOREM 2.1. *Suppose* $\mathfrak{A}$ *is a model for* L, $\phi(x)$ *a formula of* L, *and* $\aleph_0 \leqq |\phi(\mathfrak{A})| < |A|$. *Then there exist models* $\mathfrak{B}$, $\mathfrak{C}$ *such that* $\mathfrak{B} \prec \mathfrak{A}$, $\mathfrak{B} \prec \mathfrak{C}$, $\phi(\mathfrak{B}) = \phi(\mathfrak{C})$, $|\phi(\mathfrak{C})| = \aleph_0$, *and* $|C| = \aleph_1$.

Proof. Let $\mathfrak{m} = |\phi(\mathfrak{A})|$. $\mathfrak{A}$ has an elementary submodel of power $\mathfrak{m}^+$ which contains $\phi(\mathfrak{A})$. Thus we may as well assume that $|A| = \mathfrak{m}^+$. Let $<$ be a well ordering of A of type $\mathfrak{m}^+$. Let $\psi(xyv_1 \cdots v_n)$ be a formula in $L(<)$ and $a_1, \cdots, a_n \in A$. We note that in the model $(\mathfrak{A}, <)$, if there are arbitrarily large $y \in A$ such that for some $x \in \phi(\mathfrak{A})$, $\psi(xya_1 \cdots a_n)$ holds, then there is a fixed $x \in \phi(\mathfrak{A})$ such that for arbitrarily large $y \in A$, $\psi(xya_1 \cdots a_n)$ holds. In other words, the sentence below holds in $(\mathfrak{A}, <)$:

$$\text{(1)} \quad \begin{aligned} &\forall v_1 \cdots v_n [\forall z \exists y \exists x (z < y \wedge \phi(x) \wedge \psi(xyv_1 \cdots v_n)) \rightarrow \\ &\qquad \exists x \forall z \exists y (z < y \wedge \phi(x) \wedge \psi(xyv_1 \cdots v_n))]. \end{aligned}$$

The main step in the proof is to show the following:

(2) Every countable model $(\mathfrak{B}_0, <_0) \equiv (\mathfrak{A}, <)$ has a countable proper elementary extension $(\mathfrak{B}_1, <_1)$ such that $\phi(\mathfrak{B}_1) = \phi(\mathfrak{B}_0)$.

After (2) is established we may take a countable elementary submodel $(\mathfrak{B}, <)$ of $(\mathfrak{A}, <)$. Using (2) ω_1 times we construct an elementary chain

$$(\mathfrak{B}, <) = (\mathfrak{B}_0, <_0) \prec (\mathfrak{B}_1, <_1) \prec \cdots \prec (\mathfrak{B}_\alpha, <_\alpha) \prec \cdots, \alpha < \omega_1,$$

such that for each α, $|B_\alpha| = \aleph_0$, $B_\alpha \neq B_{\alpha+1}$, and $\phi(\mathfrak{B}_\alpha) = \phi(\mathfrak{B})$. Let $\mathfrak{C}$ be the union of the elementary chain $\mathfrak{B}_\alpha$, $\alpha < \omega_1$. Then $\mathfrak{B} \prec \mathfrak{C}$, $\phi(\mathfrak{C}) = \phi(\mathfrak{B})$, $|\phi(\mathfrak{C})| = \aleph_0$, and $|C| = \aleph_1$.

It remains to prove (2). First we make $(\mathfrak{B}_0, <_0)$ into an ω-model

$$(\mathfrak{B}_0, <_0, N, 0, 1, \cdots)$$

by letting $N = \phi(\mathfrak{B}_0)$ and choosing distinct $0, 1, \cdots$ so that $N = \{0, 1, \cdots\}$. Then add a new individual constant c_b to our language for each $b \in B_0$ and form the ω-model

$$\mathfrak{B}_0^* = (\mathfrak{B}_0, <_0, N, 0, 1, \cdots, b)_{b \in B_0}.$$

Finally we add one more new constant c. This gives us the logic L^ω with the extra predicate $<$ and constants c_b, $b \in B_0$ and c. In this logic let T be the theory consisting of the following sentences:

all sentences true in $\mathfrak{B}_0^*$;

the sentences $c_b < c$, for each $b \in B_0$.

Any finite subset of T has a model $(\mathfrak{B}_0^*, b)$ where c is interpreted by a sufficiently large element b in the ordering $<_0$. Therefore T has a model.

We now show that T is ω-complete. Observe that a sentence $\psi(c)$ is consistent with T if and only if there are arbitrarily large elements b in the ordering $<_0$ which satisfy the sentence $\psi(y)$ in $\mathfrak{B}_0^*$. In other words

$\psi(c)$ is consistent with T if and only if the sentence

(3) $$\forall z \exists y (z < y \wedge \psi(y))$$

holds in $\mathfrak{B}_0^*$.

We also note that given any formula $\psi(v_0 \cdots)$ we may find a formula $\psi'(v_0 \cdots)$, in which the symbols $N, 0, 1, \cdots$ do not appear such that $\forall v_0 \cdots (\psi \leftrightarrow \psi')$ is a consequence of T. To form ψ' we replace $N(x)$ everywhere by $\phi(x)$ and replace numerical constants m by the constants c_b which denote the same element of $\mathfrak{B}_0^*$.

Suppose now that the sentence

$$\exists x (N(x) \wedge \theta(x c_{b_1} \cdots c_{b_n} c))$$

is consistent with T. We may assume that the symbols $N, 0, 1, \cdots$ do not occur in θ. By (3) the sentence

$$\forall z \exists y \exists x (z < y \wedge N(x) \wedge \theta(x c_{b_1} \cdots c_{b_n} y))$$

holds in $\mathfrak{B}_0^*$. Then $\mathfrak{B}_0^*$ also satisfies the sentence

$$\forall z \exists y \exists x (z < y \wedge \phi(x) \wedge \theta(x c_{b_1} \cdots c_{b_n} y)).$$

Using the fact that $(\mathfrak{B}_0, <_0) \equiv (\mathfrak{A}, <)$, we see from (1) that

$$\exists x \forall z \exists y (z < y \wedge \phi(x) \wedge \theta(x c_{b_1} \cdots c_{b_n} y))$$

holds in $\mathfrak{B}_0^*$. Then for some $m < \omega$,

$$\forall z \exists y (z < y \wedge \theta(m c_{b_1} \cdots c_{b_n} y))$$

holds in $\mathfrak{B}_0^*$. Using (3) again, we see that the sentence

$$\theta(m, c_{b_1} \cdots c_{b_n} c)$$

is consistent with T. Therefore T is ω-complete. By the ω-completeness theorem T has an ω-model

$$(\mathfrak{B}_1, <_1, N, 0, 1, \cdots, b, c)_{b \in B_0}.$$

$(\mathfrak{B}_1, <_1)$ is an elementary extension of $(\mathfrak{B}_0, <_0)$. It is proper because $c \in B_1 - B_0$. Moreover, $N = \phi(\mathfrak{B}_1) = \phi(\mathfrak{B}_0)$. This proves (2).

As a corollary we get a two-cardinal theorem for PC_δ^ω classes.

Corollary 2.2. *Let $\mathfrak{A}$ be a model for L, $\phi(x)$ a formula in L, and $\aleph_0 \leqq |\phi(\mathfrak{A})| < |A|$ (just as in Theorem 2.1). Suppose K is a PC_δ^ω class and $\mathfrak{A} \in K$. Then there exist $\mathfrak{B}, \mathfrak{C} \in K$ such that $\mathfrak{B} \prec \mathfrak{A}, \mathfrak{B} \prec \mathfrak{C}$, $\phi(\mathfrak{B}) = \phi(\mathfrak{C})$, $|\phi(\mathfrak{C})| = \aleph_0$, and $|C| = \aleph_1$.*

Proof. For some theory T in an ω-logic, L^ω $(P_0, P_1, \cdots)$, K is the class of all reducts to L of ω-models of T. We need only apply Theorem 2.1 to models of T with the formula $\psi(x) = \phi(x) \vee N(x)$ in place of $\phi(x)$. The point is that if $\mathfrak{A}^*$ is an ω-model of $T, \mathfrak{B}^* \prec \mathfrak{A}^*, \mathfrak{B}^* \prec \mathfrak{C}^*$, and $\psi(\mathfrak{B}^*) = \psi(\mathfrak{C}^*)$, then $\mathfrak{B}^*$ and $\mathfrak{C}^*$ are also ω-models of T.

Morley in [9] proved a (quite different) two-cardinal theorem in which the formula $\phi(x)$ is an infinite conjuction of formulas of Linstead of a single formula.

Our next corollary is a generalization of Vaught's two-cardinal theorem in which $\phi(x)$ can be a formula in $L_{\omega_1\omega}$ (this includes as a special case the infinite conjuctions $\phi(x)$ of formulas of L).

Corollary 2.3. *Let $\mathfrak{A}$ be a model for $L, \phi(x)$ a formula in $L_{\omega_1\omega}$, and $\aleph_0 \leqq |\phi(\mathfrak{A})| < |A|$. Then there exist models $\mathfrak{B}, \mathfrak{C}$ such that $\mathfrak{B} \prec \mathfrak{A}, \mathfrak{B} \prec \mathfrak{C}$, $\phi(\mathfrak{B}) = \phi(\mathfrak{C})$, $|\phi(\mathfrak{C})| = \aleph_0$, and $|C| = \aleph_1$. Moreover, if K is a PC_δ^ω class and $\mathfrak{A} \in K$, then we may also choose $\mathfrak{B}, \mathfrak{C}$ so that $\mathfrak{B} \in K, \mathfrak{C} \in K$.*

Proof. Add an extra unary predicate symbol U to L. Let K' be the class of all models $(\mathfrak{B}, V)$ for $L(U)$ such that $\mathfrak{B} \equiv \mathfrak{A}$ and $V = \phi(\mathfrak{B})$ (and, if a K is given, $\mathfrak{B} \in K$). $V = \phi(\mathfrak{B})$ is true if and only if $(\mathfrak{B}, V)$ satisfies the sentence

$$\forall x (U(x) \leftrightarrow \phi(x))$$

of $L_{\omega_1\omega}$. It follows that K' is PC_δ^ω. Now apply Corollary 2.2 to the class K' with $U(x)$ in place of $\phi(x)$.

Corollary 2.2 is not in general true if we allow classes K which are defined using theories in a logic with uncountably many extra predicates. For instance, we could take the K which consists of those models $\mathfrak{A}$ for which $|\phi(A)| \geqq \aleph_1$.

Moreover, using an example is Scott [12] one can easily find an uncountable set Σ of sentences in $L_{\omega_1\omega}$ such that Corollary 2.2 is false for the class K of all models of Σ.

As a last corollary, we apply the proof of Theorem 2.1 to give a necessary and sufficient condition for a theory in ω-logic to have an ω-model of power $\aleph_1$.

COROLLARY 2.4. *Let T_1 be the theory in the logic $L^\omega(<)$ consisting of the following sentences:*

$$\text{“} < \textit{is a strict linear ordering}\text{”};$$

$$\forall x \exists y (x < y);$$

every sentence of the form

$$\forall v_1 \cdots \forall v_n [\forall z \exists y \exists x (z < y \wedge N(x) \wedge \psi(xyv_1 \cdots v_n)) \rightarrow$$
$$\exists x \forall z \exists y (z < y \wedge N(x) \wedge \psi(xyv_1 \cdots v_n))].$$

A theory T in L^ω has an ω-model of power $\aleph_1$ if and only if the theory $T \cup T_1$ in $L^\omega(<)$ has an ω-model.

Proof. As in the proof of Theorem 2.1, one can show that every countable ω-model of $T \cup T_1$ has a proper elementary extension which is an ω-model. It follows that $T \cup T_1$ has an ω-model of power $\aleph_1$, and the reduct of this model to L^ω is an ω-model of T of power $\aleph_1$.

Furken [3] gave an improvement of Vaught's theorem in which the formula $\phi(x,y)$ is allowed to have the extra parameter y. We shall extend Fuhrken's theorem to ω-logic. We need the following well-known generalization of the ω-completeness theorem:

If T is consistent and ω-complete with respect to $(N_i, 0_i, 1_i, \cdots)$ for $i = 0,1,2,\cdots$, then T has a model which is an ω-model with respect to each $(N_i, 0_i, 1_i, \cdots)$.

If $b \in A$, and $\phi(x,y)$ is a formula, we shall let $\phi(\mathfrak{A}, b) = \{a \in A : (a,b)$ satisfies $\phi(x,y)$ in $\mathfrak{A}\}$.

THEOREM 2.5. *Suppose K is a PC_δ^ω class, $\phi(x,y)$ is a formula of L (or of $L_{\omega_1\omega}$), and $\mathfrak{A}$ is a model in K whose power $|A|$ is regular and such that*

$$\text{(i)} \qquad |\phi(\mathfrak{A}, b)| < |A| \textit{ for all } b \in A.$$

Then K contains a model $\mathfrak{C}$ of power $\aleph_1$ such that

$$\text{(ii)} \qquad |\phi(\mathfrak{C}, c)| \leqq \aleph_0 \textit{ for all } c \in C.$$

Proof. It suffices to prove that for every ω-model $\mathfrak{A}$ of regular power and with the property (i), there is an ω-model $\mathfrak{C} \equiv \mathfrak{A}$ of power $\aleph_1$ with the property (ii). We argue as in the proof of Theorem 2.1. Let $<$ be a well ordering of A of type $|A|$. Since $|A|$ is regular, the model $(\mathfrak{A}, <)$ satisfies the formula (1) with $\phi(x, v_1)$ in place of $\phi(x)$. Let $(\mathfrak{B}_0, <_0) \equiv (\mathfrak{A}, <)$ be a countable ω-model. For each $b \in B_0$, let

$$N_b = \phi(\mathfrak{B}_0, b) = \{0_b, 1_b, \cdots\}.$$

Using the ω-completeness theorem with respect N and N_b for each $b \in B_0$, we see that $(\mathfrak{B}_0, <_0)$ has a proper elementary extension $(\mathfrak{B}_1, <_1)$ which is an ω-model such that

$$\phi(\mathfrak{B}_0, b) = \phi(\mathfrak{B}_1, b) \text{ for all } b \in B_0.$$

By iterating this construction ω_1 times we obtain the desired model $\mathfrak{C}$.

Notice that Theorem 2.5 is a corollary of Theorem 2.1 when the power of $\mathfrak{A}$ is a successor cardinal, but not when the power of $\mathfrak{A}$ is an inaccessible cardinal. Fuhrken [3] gave an example showing that his theorem fails when the power of $\mathfrak{A}$ is singular, and it follows that our Theorem 2.5 also fails in that case.

3. **Homogeneous theories.** Consider a model $\mathfrak{A}$ for L of uncountable power $\mathfrak{m}$. We say that $\mathfrak{A}$ is *homogeneous* if for any two elementary submodels $\mathfrak{B}, \mathfrak{C} \prec \mathfrak{A}$ of power less than $\mathfrak{m}$, any isomorphism of $\mathfrak{B}$ onto $\mathfrak{C}$ can be extended to an automorphism of $\mathfrak{A}$. This notion is due to Morley and Vaught [10] and is based on earlier work of Jónsson [4]. A more useful equivalent definition of homogeneous is given in the following lemma of Morley and Vaught [10].

LEMMA 3.1. *A model $\mathfrak{A}$ for L of power $\mathfrak{m}$ is homogeneous if and only if the following holds.*

For all $\alpha < \mathfrak{m}$ and all $a, b \in A^\alpha$, if

$$(\mathfrak{A}, a) \equiv (\mathfrak{A}, b)$$

then for all $c \in A$ there exists $d \in A$ such that

$$(\mathfrak{A}, a, c) \equiv (\mathfrak{A}, b, d).$$

The next lemma is an observation of Morley.

LEMMA 3.2. *Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are homogeneous. Then $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic if and only if*

(1) $|A| = |B|$;

(2) *For every finite sequence a in A there is a finite sequence b in B such that*

$$(\mathfrak{A}, a) \equiv (\mathfrak{B}, b),$$

and vice versa.

Proof. If $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic then (1) and (2) obviously hold.

Assume (1) and (2). Using Lemma 3.1, one can show by induction on α that for each $\alpha < \mathfrak{m}$ and $a \in A^\alpha$ there exists $b \in B^\alpha$ such that $(\mathfrak{A}, a) \equiv (\mathfrak{B}, b)$, and vice versa. By a "back and forth" argument it follows that $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic.

We now prove the main theorem of this section.

THEOREM 3.3. *Suppose that K is a PC_δ^ω class. If every model $\mathfrak{A} \in K$ of power $\aleph_1$ is homogeneous, then every uncountable model in K is homogeneous.*

Proof. Let K contain a model $\mathfrak{A}$ of power $\mathfrak{m} \geqq \aleph_1$ which is not homogeneous. We shall find a model $\mathfrak{C} \in K$ of power $\aleph_1$ which is not homogeneous. By Lemma 3.1 there exists $\alpha < \mathfrak{m}, a, b \in A^\alpha$, and $c \in A$ such that $(\mathfrak{A}, a) \equiv (\mathfrak{A}, b)$ but there is no

$d \in A$ with

$$(\mathfrak{A}, a, c) \equiv (\mathfrak{A}, b, d).$$

Now let P be a new binary predicate, let

$$R = \{\langle \alpha_\beta, b_\beta \rangle : \beta < \alpha\},$$

let $\phi(x)$ be the formula

$$\exists y P(xy) \vee \exists y P(yx),$$

and consider the model $(\mathfrak{A}, R)$. We have

$$\phi((\mathfrak{A}, R)) = \text{range}(a) \cup \text{range}(b),$$

and hence

$$|\phi((\mathfrak{A}, R))| < \mathfrak{m}.$$

Let

$$\psi_0(v_0),\ \psi_1(v_0 v_1),\ \psi_2(v_0 v_1 v_2), \cdots$$

be a list of all formulas of L. The list is chosen in such a way that all the free variables of ψ_n are among $v_0, \cdots, v_n$. Then $(\mathfrak{A}, R)$ satisfies the following sentence θ of $L_{\omega_1 \omega}(P)$:

$$\exists x_0 \forall y_0 \bigvee_{n < \omega} \exists x_1 \cdots x_n y_1 \cdots y_n$$

$$[P(x_1 y_1) \wedge \cdots \wedge P(x_n y_n) \wedge \neg(\psi_n(x_0 \cdots x_n) \leftrightarrow \psi_n(y_0 \cdots y_n))].$$

Let K' be the class of all models $(\mathfrak{B}, S)$ of θ such that $\mathfrak{B} \in K$. Then K' is PC_δ^ω and $(\mathfrak{A}, R) \in K'$. By Corollary 2.2 there is a model $(\mathfrak{C}, S) \in K'$ such that $(\mathfrak{C}, S) \equiv (\mathfrak{A}, R)$, $|\phi((\mathfrak{C}, S))| = \aleph_0$, and $|C| = \aleph_1$. Let $\langle a_0' b_0' \rangle, \langle a_1' b_1' \rangle, \cdots$ be a list of all pairs $\langle c, d \rangle \in S$. The list is countable because $|\phi((\mathfrak{C}, S))| = \aleph_0$. It follows from $(\mathfrak{C}, S) \equiv (\mathfrak{A}, R)$ that

$$(\mathfrak{C}, a') \equiv (\mathfrak{C}, b').$$

Since $(\mathfrak{C}, S)$ satisfies θ, there is a $c' \in C$ such that for no $d' \in C$ does

$$(\mathfrak{C}, a', c') \equiv (\mathfrak{C}, b', d').$$

Hence $\mathfrak{C}$ is a model in K of power $\aleph_1$ which is not homogeneous.

Theorem 3.3 above seems to give a new result even for first order logic:

COROLLARY 3.4. *If T is a theory in L and every model of T of power $\aleph_1$ is homogeneous then every uncountable model of T is homogeneous.*

In Theorem 3.3, or even in the special case 3.4, we cannot replace $\aleph_1$ by an arbitrary uncountable cardinal. For Morley has given an ingenious example of a theory T in L such that all models of T of power at least $2^{2^{\aleph_0}}$ are homogeneous but T has models of each power less than $2^{2^{\aleph_0}}$ which are not homogeneous.

4. Categorical theories. In this section we shall give a relatively short new proof of the upward part of Morley's categoricity theorem (using the continuum hypothesis). Actually we shall prove a more general result in Theorem 4.1, involving PC_δ^ω classes. Theorem 4.1 does not use the continuum hypothesis — it comes in only when we derive the upward part of Morley's theorem in Corollary 4.2.

We shall say that a class K of models is *categorical in power* $\mathfrak{m}$ if all models in K of power $\mathfrak{m}$ are isomorphic. We allow the possibility that K has no members of power $\mathfrak{m}$.

THEOREM 4.1. *Suppose K is a PC_δ^ω class, K is categorical in power $\aleph_1$ and K contains a homogeneous model of power $\aleph_1$. Then K is categorical in every uncountable power.*

Proof. By Theorem 3.4, all uncountable models in K are homogeneous. Suppose $\mathfrak{A}, \mathfrak{B} \in K$, both $\mathfrak{A}, \mathfrak{B}$ have power $\mathfrak{m} \geqq \aleph_1$, and $\mathfrak{A}, \mathfrak{B}$ are not isomorphic. Then by Lemma 3.2 there is, say, a finite sequence $a_1, \cdots, a_n \in A$ such that for no $b_1, \cdots, b_n \in B$ do we have

$$(\mathfrak{A}, a_1, \cdots, a_n) \equiv (\mathfrak{B}, b_1, \cdots, b_n).$$

Choose $\mathfrak{A}_0 \prec \mathfrak{A}$, $\mathfrak{B}_0 \prec \mathfrak{B}$ such that $\mathfrak{A}_0, \mathfrak{B}_0 \in K$, $|A_0| = |B_0| = \aleph_1$, and $a_1, \cdots, a_n \in A_0$. Then B_0 has no finite sequence $b_1, \cdots, b_n$ such that

$$(\mathfrak{A}_0, a_1, \cdots, a_n) \equiv (\mathfrak{B}_0, b_1, \cdots, b_n).$$

Hence $\mathfrak{A}_0$ and $\mathfrak{B}_0$ cannot be isomorphic. This contradicts our assumption that K is categorical in power $\aleph_1$. Therefore K is categorical in all powers $\mathfrak{m} \geqq \aleph_1$.

COROLLARY 4.2. *Assume the continuum hypothesis $2^{\aleph_0} = \aleph_1$. Suppose K is a PC_δ class and K is categorical in power $\aleph_1$. Then K is categorical in every uncountable power.*

Proof. It is shown in Morley and Vaught [10] that, if $2^{\aleph_0} = \aleph_1$, then every PC_δ class which has infinite models has a homogeneous model of power $\aleph_1$.

The author first proved 4.1 in a way similar to the proof of 3.3 without using Lemma 3.2. We are indebted to Morley for pointing out that the proof could be simplified by using Lemma 3.2.

For some examples of theories T in L which are categorical in power $\aleph_1$ see Morley [8]. In each known example, it happens to be obvious that T is categorical in every power $\mathfrak{m} \geqq \aleph_1$, because the proof that T is categorical in power $\aleph_1$ also works for all $\mathfrak{m} > \aleph_1$. The same thing happens for each known example of a PC_δ^ω class K which is categorical and has a homogeneous model of power $\aleph_1$.

The following are examples of PC_δ classes K which are categorical in power $\aleph_1$ but cannot be characterized as the class of all models of some theory T in L.

1. The class of all models $\langle A, E\rangle$ where E is an equivalence relation over A, each equivalence class has power $|A|$, and there are $|A|$ different equivalence classes.

2. The class of all models $\langle A, E_1, E_2\rangle$ where E_1, E_2 are both equivalence relations with $|A|$ equivalence classes and each equivalence class of E_1 meets each equivalence of E_2 in exactly one element.

3. The class of models $\langle A, P_0, P_1, P_2, \cdots\rangle$ where $P_0, P_1, P_2, \cdots$ are disjoint subsets of A of power $|A|$, and

$$|A - \bigcup_{i<\omega} P_i| = |A|.$$

4. The class of abelian groups $\langle A, +\rangle$ which have $|A|$ elements of order 2, order 3, and order 6, and no elements of any other order.

5. Let L' be an extension of the language L, let T be a theory in L' categorical in power $\aleph_1$, and let K be the class of all reducts to L of models of T.

We shall now list some examples of PC_δ^ω classes to which Theorem 4.1 applies. We shall give examples of PC_δ^ω classes K with the following property:

(*) There is a countable extension L' of the logic L and a PC_δ^ω class K' of models of L' such that K' is categorical and has a homogeneous model in power $\aleph_1$, and K is the class of all reducts to L of models in K'.

By Theorem 4.1, the class K' in the condition (*) is categorical in every uncountable power. It follows at once that every class K with the property (*) is a PC_δ^ω class categorical in every uncountable power.

7. For each equational class M of algebras, the class K of all free algebras in M, also the class K' of all models $(\mathfrak{A}, U)$ where $\mathfrak{A}$ is an algebra in M freely generated by U. Note that K' has a homogeneous model of power K_1.

8. The class K of all models $\langle A, E\rangle$ where E is an equivalence relation over A all of whose equivalence classes are of power $\aleph_0$. A suitable class K' is the class of all ω-models $\langle A, E, F, N, \cdots\rangle$ such that E, F are equivalence relations over A, N is one equivalence class of E, and each equivalence class of F meets each equivalence class of E in exactly one point.

9. Like 8 but all equivalence classes of E have finite power, and for each $n < \omega$ there are $|A|$ classes of power n.

10. Like 8 but all equivalence classes of E have power $|A|$ and there are $\aleph_0$ equivalence classes.

11. The class of all abelian groups $\langle A, +\rangle$ in which the order of each element is a product of distinct primes and for each prime p there are $|A|$ elements of order p.

12. The class of all trees $\langle A, <\rangle$ in which each element has finitely many predecessors and $|A|$ immediate successors (and the tree has only one root).

13. The class of all models isomorphic to a model $\langle A, \subseteq \rangle$ where A is the set of all finite subsets of a set X.

14. The class of all models $\langle A, P_0, P_1, \cdots \rangle$ where the P_i are disjoint subsets of A of power $|A|$ and $\bigcup_{i<\omega} P_i = A$.

15. Let X be a countable set of subsets of ω. The class of all models $\langle A, P_0, P_1, \cdots \rangle$ where each P_i is a subset of A, and for each $y \subset \omega$,

$$\bigcap_{i \in y} P_i \cap \bigcap_{i \notin y} (A - P_i)$$

has power $|A|$ if $y \in X$ and is empty if $y \notin X$.

16. Let F be a countable field. The class of all pure transcendental extensions of F.

17. Let $\mathfrak{A}$ be a countable algebra which has an element 0 idempotent for all the operations of $\mathfrak{A}$. The class of all weak direct powers $\mathfrak{A}^I$ (whose universe set is the set of all functions $f \in A^I$ with $f(i) = 0$ for all but finitely $i \in I$). Note that 4, 1 are special cases of 17.

18. Let $\mathfrak{A}$ be a countable model for L. The class of all cardinal multiples $\mathfrak{A}I$ of $\mathfrak{A}$ (unions of disjoint copies $\mathfrak{A}_i$, $i \in I$, of $\mathfrak{A}$).

19. Let $K_0, K_1, \cdots$ all have the property $(*)$, and let K be the class of all cardinal sums

$$\mathfrak{A}_0 + \mathfrak{A}_1 + \cdots$$

of models $\mathfrak{A}_n \in K_n$ all of the same power.

20. Let K_0 have property $(*)$ and let K be the class of cardinal multiples $\mathfrak{A}I$, where $\mathfrak{A} \in K_0$ and $|A| = |I|$.

The following problem is open: Generalize the results of this paper by replacing $\aleph_1$ by an arbitrary uncountable cardinal. For example, are the following three things true for all $\mathfrak{m} \geqq \aleph_1$ and all PC_δ^ω classes K, or if not, what else must be assumed about K or $\mathfrak{m}$?

A? If there is an $\mathfrak{A} \in K$ with

$$\mathfrak{m} = |\phi(\mathfrak{A})| < |A|,$$

and if $\aleph_1 \leqq n \leqq \mathfrak{m}$, there is a $\mathfrak{B} \in K$ such that

$$\mathfrak{n} = |\phi(\mathfrak{B})| < |B|.$$

B? If every model of K of power $\mathfrak{m}$ is homogeneous, then every model of K of power greater than $\mathfrak{m}$ is homogeneous.

C? If K is categorical in power $\mathfrak{m}$ then K is categorical in every power greater than $\mathfrak{m}$.

D? Suppose that T is a complete theory in L and every model of T of power $\aleph_1$ is homogeneous. Does it follow that T is categorical in power $\aleph_1$?

The referee has pointed out that the answer to question A is negative when $\mathfrak{m} = \aleph_\omega$ (assuming the GCH), because of an example due to Chang. By combining Chang's example with a result of Morley [9], we see that question A has a negative answer whenever $\mathfrak{m}$ is a singular cardinal of cofinality less than $\aleph_{\omega_1}(GCH)$. To make question A reasonable, we must stipulate either that $\mathfrak{m}$ is regular or that $\mathfrak{m}$ has cofinality at least $\aleph_{\omega_1}$.

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